Journal of Geometry and Physics 9 (1992) 149–154
North-Holland

JOURNAL OF GEOMETRY AND PHYSICS

Pseudo-isotropic Lorentzian hypersurfaces in Minkowski space

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Received 6 May 1991
(Revised 3 October 1991)

We show that a Lorentzian hypersurface M^n of the Minkowski space $\mathbb{R}^{n+1}_1$ is pseudo-isotropic if and only if M^n is flat and minimal. Next we obtain a classification of all pseudo-isotropic Lorentzian hypersurfaces in $\mathbb{R}^3_1$ and $\mathbb{R}^4_1$.

Keywords: *Minkowski space, hypersurfaces*
1991 MSC: *53 B 30, 53 B 25*

1. Introduction

The notion of *pseudo-isotropic* (or *pseudo-null*) submanifolds of semi-Riemannian manifolds was introduced by the second named author in the early seventies (see, e.g., refs. [2], [3, chs. 5,6,7]). Since then a lot of work has been done on this subject by, for instance, L. Vanhecke, B. Rouxel and the second and third named author.

We recall that a manifold is called *isotropic* (or *null*) if its metric tensor is degenerate. In particular, isotropic submanifolds in space–times play an important role in relativity theory. Related to this notion of isotropy, a Lorentzian hypersurface M^n in a Minkowski space $\mathbb{R}^{n+1}_1$ is called pseudo-isotropic when its Gauss map (or spherical representation—the map which sends a point p of M^n onto the endpoint of the unit normal vector $\xi(p)$ at p, translated to the origin) is null (or "ametrical"), or when its third fundamental form vanishes.

In theorem 1, we obtain that M^n is pseudo-isotropic if and only if M^n is a *flat minimal* hypersurface in $\mathbb{R}^{n+1}_1$. This result holds for all dimensions n. The "only if" part already appeared in the literature, see, for instance, refs. [2] and

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[4], or ref. [5] for a generalization to higher codimension. In theorem 2 and theorem 3, for the particular cases of dimension $n = 2$ or $n = 3$, we obtain full classification results of all pseudo-isotropic Lorentzian hypersurfaces in $\mathbb{R}^3_1$ and $\mathbb{R}^4_1$. For an introduction to semi-Riemannian geometry, the reader is referred to ref. [1].

2. Definitions and statements of results

Let $\mathbb{R}^{n+1}_1$ be the $(n+1)$-dimensional Minkowski space, and let M^n be a Lorentz hypersurface of $\mathbb{R}^{n+1}_1$, i.e., the induced metric on M^n has signature $(n-1, 1)$. We denote both the Lorentz metric on $\mathbb{R}^{n+1}_1$ and the induced metric on M^n by $\langle\ ,\ \rangle$. Let ∇ be the Levi-Civita connection of M^n and $\tilde{\nabla}$ the Levi-Civita connection of $\mathbb{R}^{n+1}_1$. As usual, a vector v is called a null-vector if $\langle v, v\rangle = 0$. Let S denote the shape operator of M^n. Then M^n is called pseudo-isotropic if $\langle SX, SY\rangle = 0$ for all tangent vectors X and Y. In other words, M^n is pseudo-isotropic if SX is a null-vector for every X. Since S is symmetric, this is equivalent to $S^2 = 0$. For hypersurfaces we prove the following theorem.

Theorem 1. *A Lorentzian hypersurface M^n of $\mathbb{R}^{n+1}_1$ is pseudo-isotropic if and only if M^n is flat and minimal.*

For $n = 2$ we can give a complete classification of pseudo-isotropic hypersurfaces in the following way.

Theorem 2. *A Lorentzian hypersurface M^2 of $\mathbb{R}^3_1$ is pseudo-isotropic if and only if there is an open dense subset of M^2 such that each connected component is a part of a null-cylinder on a null-curve.*

A null-curve in $\mathbb{R}^3_1$, or in any Minkowski space $\mathbb{R}^m_1$, is a curve such that its tangent vector is a null-vector at any point. If α is a curve, and b is a constant vector in $\mathbb{R}^3_1$ which is nowhere tangent to α, then $x(s,t) = \alpha(s) + tb$ is a cylinder on α and it is called a null-cylinder if b is a null-vector.

Theorem 3. *A Lorentzian hypersurface M^3 of $\mathbb{R}^4_1$ is pseudo-isotropic if and only if there is an open dense subset of M^3 such that each connected component U is a part of (1) the product of a null-cylinder on a null-curve which lies in a Lorentzian subspace $\mathbb{R}^3_1$ of $\mathbb{R}^4_1$ and $\mathbb{R}$, the orthogonal complement of $\mathbb{R}^3_1$, i.e., $U \subset M^2 \times \mathbb{R} \subset \mathbb{R}^3_1 \times \mathbb{R} = \mathbb{R}^4_1$, or (2) the osculating hypersurface of a null-curve, or (3) up to a translation, a cone centered at the origin on the tangential surface of a curve on the null-cone.*

The osculating hypersurface of a curve α in $\mathbb{R}^4_1$ is the hypersurface given by the parametrization $x(s,t_1,t_2) = \alpha(s) + t_1\alpha'(s) + t_2\alpha''(s)$, whereby α', α'' and α''' are assumed to be linearly independent. The tangential surface of a curve α in $\mathbb{R}^4_1$ is the surface given by $x(s,t) = \alpha(s) + t\alpha'(s)$. Here α' and a'' have to be linearly independent. Note that the tangential surface of a curve on the null-cone is isotropic and that a cone, say centered at the origin, on a surface M^2 is a three-dimensional hypersurface only if the position vector is never tangent to M^2.

Both in theorem 2 and theorem 3 the expression "open dense subset" appears. This comes in because in the proofs, whenever we need the fact that some function is not zero, we will always restrict to an open part where this function is not zero, and to the interior of the set where this function vanishes. Details are left for the reader.

3. Proof of the theorems

3.1. PROOF OF THEOREM 1

If M^n is pseudo-isotropic in $\mathbb{R}^{n+1}_1$, then SX is a null-vector for all X. Hence $\mathrm{im}(S)$ is a part of the null-cone of T_pM at any point $p \in M$. Since $\mathrm{im}(S)$ is a linear subspace, it is a line. So $\mathrm{rank}(S) \leq 1$, which implies that M^n is flat.

Now fix a point $p \in M$. If $S_p = 0$, then clearly $\mathrm{trace}(S) = 0$. So we can assume that $S_p \neq 0$. Since $\mathrm{im}(S)$ is a line, there is a nonzero vector Z and a nonzero one-form α such that $S(X) = \alpha(X)Z$ for all X. Moreover, Z is a null-vector. Since the shape operator is symmetric w.r.t. $\langle\ ,\ \rangle$, we obtain that

$$\alpha(X)\langle Z,Y\rangle = \alpha(Y)\langle Z,X\rangle$$

for all X and Y. From this equation it is easy to obtain that $\ker(S) = \ker(\alpha) = Z^{\perp}$, where $Z^{\perp} = \{X \mid \langle X,Z\rangle = 0\}$.

Now we choose a basis $\{e_1,e_2,\ldots,e_n\}$ of T_pM such that $\langle e_1,e_1\rangle = 0$, $\langle e_1,e_2\rangle = 1$, $\langle e_2,e_2\rangle = 0$, $e_2 = Z$, $\langle e_1,e_i\rangle = \langle e_2,e_i\rangle = 0$ and $\langle e_i,e_j\rangle = \delta_{ij}$ for $i,j > 2$. But then we have that $S(e_1) = \alpha(e_1)e_2$ and $S(e_2) = S(e_3) = \cdots = S(e_n) = 0$. In particular we obtain $\mathrm{trace}(S) = 0$.

Conversely, if M^n is flat and minimal, then $\mathrm{rank}(S) \leq 1$ and $\mathrm{trace}(S) = 0$. Take a point $p \in M$. If $S_p = 0$, then clearly $S_p^2 = 0$. So we assume that $S_p \neq 0$. So again there is a nonzero vector Z and a nonzero one-form α such that $S(X) = \alpha(X)Z$ for all X. Like above, we know that $\ker(S) = \ker(\alpha) = Z^{\perp}$. We have to show that Z is a null-vector, i.e. that $Z \in Z^{\perp}$. If Z is not a null-vector, then Z is an eigenvector of S with nonzero eigenvalue. Since zero is an eigenvalue of S with multiplicity $n-1$, $\mathrm{trace}(S) \neq 0$. This is a contradiction. □

3.2. PROOF OF THEOREM 2

Let M^2 be a pseudo-isotropic surface in $\mathbb{R}^3_1$, immersed by x. By theorem 1, we know that M^2 is flat and minimal. So around each point there exist coordinates $\{u, v\}$ such that $\langle x_u, x_u\rangle = \langle x_v, x_v\rangle = 0$ and $\langle x_u, x_v\rangle = 1$. Then $\operatorname{trace}(S) = 2\langle S(x_u), x_v\rangle = 0$ and $\det(S) = -\langle S(x_u), x_u\rangle\langle S(x_v), x_v\rangle = 0$. Hence we can assume that $\langle S(x_u), x_u\rangle = 0$. Therefore $S(x_u) = 0$. But then $\widetilde{\nabla}_{x_u} x_u = 0$ and $\widetilde{\nabla}_{x_v} x_u = 0$, so that x_u is a constant null-vector b on M^2. Let β be any v-line. Then M^2 is a null-cylinder on the null-curve β.

Conversely, let $x(u, v) = \beta(u) + vb$ be a null-cylinder on a null-curve β. We can assume that $\langle \beta'(u), b\rangle = 1$, by taking another parametrization of β if necessary. Then $\langle x_u, x_u\rangle = \langle x_v, x_v\rangle = 0$ and $\langle x_u, x_v\rangle = 1$. Since $x_{uu} = 0$, we immediately obtain that M^2 is flat and minimal. □

3.3. PROOF OF THEOREM 3

Let M^3 be a pseudo-isotropic surface in $\mathbb{R}^4_1$. By theorem 1, we know that M^3 is flat and minimal. We assume that S is not zero. Then $\ker(S_p)$ is a two-dimensional linear subspace of the tangent space at any point p. From the proof of theorem 1 we also know that $\operatorname{im}(S) \subset \ker(S)$. The distribution $p \mapsto \ker(S_p)$ is known to be totally geodesic, so that each integral manifold is mapped into a two-dimensional affine subspace of $\mathbb{R}^4_1$. Now it is clear that we can choose a curve α on M^3 and vector fields E_1 and E_2 along α, spanning $\ker(S)$ such that $\langle \alpha', \alpha'\rangle = \langle E_1, E_1\rangle = \langle E_1, E_2\rangle = \langle \alpha', E_2\rangle = 0$ and $\langle \alpha', E_1\rangle = \langle E_2, E_2\rangle = 1$. Then $x(t, u, v) = \alpha(t) + uE_1(t) + vE_2(t)$ parametrizes M^3. Since $x_{ut} = E_1'(t)$ and $S(x_u) = 0$, E_1' is tangent to M^3, and the same is true for $E_2'(t)$. Hence

$$\begin{aligned} E_1' &= \langle E_1', \alpha'\rangle E_1 + \langle E_1', E_2\rangle E_2, \\ E_2' &= \langle E_2', E_1\rangle \alpha' + \langle E_2', \alpha'\rangle E_1. \end{aligned} \tag{1}$$

Case 1: $\langle E_1', E_2\rangle = 0$. In this case, both $E_1'(t)$ and $E_2'(t)$ are proportional to $E_1(t)$. This implies that $E_1(t)$ always points in a fixed null-direction F_1 and that there exists a function λ such that $E_2(t) = \lambda(t)F_1 + F_2$, where F_2 is a constant vector of length 1 and orthogonal to F_1. Then $y(t, u, v) = \alpha(t) + uF_1 + vF_2$ is a new parametrization of M^3. Let $\beta(t) = \alpha(t) + u(t)F_1 + v(t)F_2$ be a curve on M^3. Then

$$\begin{aligned} \langle \beta'(t), \beta'(t)\rangle &= 2u'(t)\langle \alpha'(t), F_1\rangle + 2v'(t)\langle \alpha'(t), F_2\rangle + v'(t)^2, \\ \langle \beta'(t), F_2\rangle &= \langle \alpha'(t), F_2\rangle + v'(t). \end{aligned}$$

We can choose the functions u and v such that $\langle \alpha'(t), F_2\rangle + v'(t) = 0$ and $2u'(t)\langle \alpha'(t), F_1\rangle = \langle \alpha'(t), F_2\rangle^2$. Then β is a null-curve, and $\beta'(t)$ is always orthogonal to F_2. So the null-cylinder $\beta(t) + uF_1$ on the null-curve β lies in the Lorentzian subspace through $\beta(0)$ orthogonal to F_2. This finishes the first case.

Case 2: $\langle E_1', E_2\rangle \neq 0$. In this case, from (1), we can write $\alpha' = f_1E_2' + f_2E_1$ for some functions f_1 and f_2. If we define a curve β on M^3 by $\beta(t) = \alpha(t) - f_1E_2$, then

$$\beta' = -f_1'E_2 + f_2E_1.$$

Using (1), this implies that there exist functions f_3 and f_4 such that

$$\beta' = f_3E_1' + f_4E_1.$$

Defining a curve γ by $\gamma(t) = \beta(t) - f_3(t)E_1(t)$, we obtain that γ' is proportional to E_1 at every point. In particular, γ is a null-curve. Furthermore, (1) implies that also γ'' lies in the plane $E_1 \wedge E_2$. If moreover $\gamma' \neq 0$, then $\gamma' \wedge \gamma'' = E_1 \wedge E_2$, so we have an osculating hypersurface on a null-curve.

If $\gamma' = 0$, then M^3 is a cone, i.e., the plane $E_1 \wedge E_2$ goes through a fixed point, say the origin of $\mathbb{R}^4_1$. Hence M^3 can be parametrized by $x(t,u,v) = uE_1(t) + vE_2(t)$. But from (1) and the assumption $\langle E_1', E_2\rangle \neq 0$ we see that the planes $E_1 \wedge E_2$ and $E_1 \wedge E_1'$ coincide. Hence M^3 can be parametrized by $x(t,u,v) = uE_1(t) + vE_1'(t)$. So M^3 is a cone on the surface given by $E_1(t) + vE_1'(t)$. Since $E_1(t)$ is a null-vector for each t, this surface is the tangential surface of a curve on the null-cone.

In order to prove the converse it is sufficient to show that the hypersurfaces of types (2) and (3) are flat and minimal.

First let γ be a null-curve, such that γ', γ'' and γ''' are linearly independent at any point. Since $\langle\gamma', \gamma'\rangle = 0$ and $\langle\gamma', \gamma''\rangle = 0$, we have that $\langle\gamma'', \gamma''\rangle > 0$, and we can parametrize γ such that $\langle\gamma'', \gamma''\rangle = 1$. Then $\langle\gamma', \gamma'''\rangle = -1$. Consider the osculating hypersurface $x(t,u,v) = \gamma(t) + u\gamma'(t) + v\gamma''(t)$. Then clearly γ''' is tangent and $\ker(S) = \gamma' \wedge \gamma''$, so that the hypersurface is flat. In order to show that it is minimal, we have to show that $S\gamma'''$ lies in $\gamma' \wedge \gamma''$, i.e., $\langle S\gamma''', \gamma'\rangle = 0$. But this follows immediately from the symmetry of S.

Secondly, let γ be a curve on the null-cone, such that γ' and γ'' are linearly independent at any point. As in the first case, we can parametrize γ such that $\langle\gamma', \gamma'\rangle = 1$. Consider the cone, centered at the origin, on the tangential surface of γ. So we assume that γ, γ' and γ'' are linearly independent at any point. A parametrization can be given by $x(t,u,v) = u\gamma(t) + v\gamma'(t)$. As in the first case, one then can show that $\ker(S) = \gamma \wedge \gamma'$, so that the hypersurface is flat, and that it is minimal. □

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